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On a fractional advection dispersion equation in $\mathbb{R}^N$ involving a critical nonlinearity

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Abstract

In this paper, by using the variational principle of Ekeland, we prove the existence of at least one solution to the fractional advection dispersion equation in $\mathbb{R}^N$.

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1 Introduction

The aim of this paper is to establish the existence of nontrivial solutions to the following fractional advection dispersion equation

$$\begin{cases} h\left(-\frac{\cos(\pi\alpha)}{2}\int_{|\theta|=1}\|D_\theta^\alpha u\|_{L^2}^2 M_1(d\theta)+\frac{1}{2}\int_{\mathbb{R}^N} b(x)u^2(x)dx\right) \\ \quad\times\left(-\int_{|\theta|=1} D_\theta D_\theta^\beta u M_1(d\theta)+b(x)u(x)\right)-\int_{|\theta|=1} D_\theta D_\theta^\beta u M_2(d\theta)=f(x,u(x))+|u|^{2^*-2}u, \quad x\in\mathbb{R}^N, \\ u\in H^\alpha(\mathbb{R}^N), \end{cases} \tag{1}$$

where $N>1$, $\inf_{x\in\mathbb{R}^N} b(x)>0$, $\beta\in(0,1)$, $\alpha=\frac{\beta+1}{2}$, $M_1(d\theta)$ and $M_2(d\theta)$ are two Borel probability measures on the unit sphere in $\mathbb{R}^N$, D_θ^β denotes directional fractional derivative of order β in the direction of the unit vector θ, $2^*=\frac{2N}{N-2\alpha}$ and the functions $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ and $h:(0,+\infty)\to(0,+\infty)$ are continuous.

If $M_2\equiv 0$, then the problem (1) is related to the stationary analogue of the equation

$$u_{tt}+h\left(-\frac{\cos(\pi\alpha)}{2}\int_{|\theta|=1}\|D_\theta^\alpha u\|_{L^2}^2 M_1(d\theta)+\frac{1}{2}\int_{\mathbb{R}^N} b(x)u^2(x)dx\right)$$
$$\times\left(-\int_{|\theta|=1} D_\theta D_\theta^\beta u M_1(d\theta)+b(x)u(x)\right)=f(x,t),$$

proposed by Kirchhoff [1] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string during the vibration. The reader is referred to [1, 2, 3, 4, 5, 6] and the references therein for

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previous work on this subject. In particular, these papers discuss the historical development of the problem as well as describe situations that can be realistically modeled by (1) with a nonconstant h.

The existence of solutions to the problem (1) on bounded domain has been studied by several authors (see e.g. [7, 8, 9, 10, 11, 12]). For example, Ervin and Roop [7], investigated the variational solution to the steady state advection dispersion equation on bounded domain in $\mathbb{R}^N$, defined by

$$-\int_{|\theta|=1} D_\theta a D_\theta^{-\beta} D_\theta u M(d\theta) + b\cdot\nabla u + cu = f$$

where $0 \leq \beta < 1$, $b(x,y)$ is the velocity of the fluid, $c(x,y)u$ represents a reaction-absorption term, f is a source term, a is the diffusivity coefficient, $M(d\theta)$ is a probability density function on the unit sphere in $\mathbb{R}^d$, D_θ is the directional derivative in the direction of the unit vector θ, and $D_\theta^{-\beta}$ is the β order fractional integral. When $N=1$, $b(x)=0$, M is atomic with $M(-1)=M(1)=\frac{1}{2}$, $h\equiv 1$, $M_2\equiv 0$ without the term $|u|^{2^*-2}u$, for the problem (1), the authors in [8, 10, 11], studied the existence of solutions to the problem on bounded domain $[0,T]$, by applying the Mountain Pass theorem.

Also, in [13], the authors considered the existence of solution of (1) on $\mathbb{R}^N$ when $N>1$, $h\equiv 1$ and without the term $|u|^{2^*-2}u$, the main tools are Mountain Pass theorem and iterative technique

Inspired by the above articles, in this paper, by using the Ekeland variational principle ([14]), we would like to investigate the existence of solution to problem (1).

The paper is organized as follows. In Section 2, we give preliminary facts and provide some basic properties which are needed to prove our main result. In Section 3, we give our main result.

2 Preliminaries

In this section, we present some preliminaries and lemmas that are useful to the proof to the main results. For the convenience of the reader, we also present here the necessary definitions.

Let $(X, \|\cdot\|_X)$ be a Banach space, $(X^*, \|\cdot\|_{X^*})$ be its topological dual, and $\varphi : X \to \mathbb{R}$ be a functional. First, we recall the definition of the Palais-Smale condition which plays an important role in our paper.

Definition 1. We say that φ satisfies the Palais-Smale condition if any sequence $(u_n)\in X$ for which $\varphi(u_n)$ is bounded and $\varphi'(u_n)\to 0$ as $n\to\infty$ possesses a convergent subsequence.

Also, for the convenience of the reader, we also present here the necessary definitions from fractional calculus theory. We refer the reader to [15].

Definition 2. Let $u:\mathbb{R}^N\to\mathbb{R}$, $\alpha>0$, θ be a unit vector in $\mathbb{R}^N$. The α-th order fractional integral in direction of θ of u is given by

$$D_\theta^{-\alpha}u(x) = \frac{1}{\Gamma(\alpha)}\int_0^{+\infty}\xi^{\alpha-1}u(x-\xi\theta)d\xi,$$

and the α-th order directional derivative in the direction of θ is defined by

$$D_\theta^{\alpha}u(x) = (\nabla\cdot\theta)^n D_\theta^{-(n-\alpha)}u(x),$$

where n denotes the smallest integer greater than α.

Definition 3. Let $u : \mathbb{R}^N \to \mathbb{R}$, $\alpha < 0(\alpha > 0)$ be given. Then the α-th order fractional integral (derivative) with respect to the measure M is defined as

$$D_M^\alpha u(x) = \int_{|\theta|=1} D_\theta^\alpha u(x) M(d\theta),$$

where $M(d\theta)$ is a probability measure on the unit sphere in $\mathbb{R}^N$.

Lemma 1. ([13, Lemma 2.3]) If $\alpha > 0$, then for $u \in C_0^\infty(\mathbb{R}^N)$, we have the following Fourier transform property

$$\mathcal{F}(D_\theta^{-\alpha} u)(\xi) = (i\xi \cdot \theta)^{-\alpha} \mathcal{F}(u)(\xi),$$
$$\mathcal{F}(D_\theta^{\alpha} u)(\xi) = (i\xi \cdot \theta)^{\alpha} \mathcal{F}(u)(\xi),$$

where

$$\mathcal{F}(u)(\xi) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-i\xi \cdot x} u(x) dx. \tag{2}$$

Let us recall that for any $\alpha > 0$, $u \in C_0^\infty(\mathbb{R}^N)$ the semi-norm

$$|u|_M = \left(\int_{|\theta|=1} \|D_\theta^\alpha u\|_{L^2}^2 M(d\theta) \right)^{\frac{1}{2}},$$

and the norm

$$\|u\|_M = \left(\|u\|_{L^2}^2 + |u|_M^2 \right)^{\frac{1}{2}}, \tag{3}$$

and let the space $J_M^\alpha(\mathbb{R}^N)$ denote the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_M$.

Note that for $u \in J_M^\alpha(\mathbb{R}^N)$, there exist $\{u_n\} \subset C_0^\infty(\mathbb{R}^N)$ such that $u_n \to u$ in $J_M^\alpha(\mathbb{R}^N)$. So $u_n \to u$ in $L^2(\mathbb{R}^N)$ and

$$\int_{|\theta|=1} \|D_\theta^\alpha u_n - D_\theta^\alpha u_m\|_{L^2}^2 M(d\theta) \to 0 \quad \text{as } n \to \infty.$$

Thus, $\|D_\theta^\alpha u_n - D_\theta^\alpha u_m\|_{L^2} \to 0$ for any $\theta \in \mathbb{S}^{N-1} = \{\theta \in \mathbb{R}^N : |\theta| = 1\}$ for θ M-a.e. in $\mathbb{S}^{N-1}$. Therefor, we can denote $D_\theta^\alpha u = \omega_\theta$ and then

$$\int_{|\theta|=1} \|D_\theta^\alpha u_n - D_\theta^\alpha u\|_{L^2}^2 M(d\theta) \to 0 \quad \text{as } n \to \infty.$$

Hence, if $u \in J_M^\alpha(\mathbb{R}^N)$, then $D_\theta^\alpha u$ exists for θ M-a.e. in $\mathbb{S}^{N-1}$.

Next, for $0 < \alpha < 1$, we give the relationship between classical fractional Sobolev space $H^\alpha(\mathbb{R}^N)$ and $J_M^\alpha(\mathbb{R}^N)$, where $H^\alpha(\mathbb{R}^N)$ is defined by

$$H^\alpha(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \ |\xi|^\alpha \mathcal{F}(u) \in L^2(\mathbb{R}^N) \right\}.$$

Now, we assume that M_1 and M_2 satisfy that:

(H) Suppose that M_1 and M_2 are positive Borel measure, $M_1(d\theta) = M_1(d(-\theta))$ and there exists constant $c_{M_1} > 0$ such that

$$\int_{|\theta|=1} |\xi \cdot \theta|^{2\alpha} M_1(d\theta) \geq c_{M_1} \quad \text{for every } \xi \in \mathbb{S}^{N-1}. \tag{4}$$

Furthermore, there is $C_{M_2} > 0$ such that for $\xi \in \mathbb{S}^{N-1}$,

$$\int_{|\theta|=1} |\xi \cdot \theta|^{2\alpha} M_2(d\theta) \leq C_{M_2} \int_{|\theta|=1} |\xi \cdot \theta|^{2\alpha} M_1(d\theta) \quad \text{for every } \xi \in \mathbb{S}^{N-1}. \tag{5}$$

We need the following Lemmas:

Lemma 2. ([13, Lemma 2.6]) If the measure M_1 satisfies (H), then the spaces $H^\alpha(\mathbb{R}^N)$ and $J^\alpha_{M_1}(\mathbb{R}^N)$ are equal and have equivalent norms.

Lemma 3. ([13, Lemma 2.7]) Assume that the measure M_1 satisfies $M_1(d\theta) = M_1(d(-\theta))$, for $\theta \in \mathbb{S}^{N-1}$. Let $\alpha > 0$, $u, v \in J^\alpha_{M_1}(\mathbb{R}^N)$. Then, for M_1-a.e. $\theta \in \mathbb{S}^{N-1}$,

$$(D^\alpha_\theta u, D^\alpha_{-\theta} u) = \cos(\pi\alpha)(D^\alpha_\theta u, D^\alpha_\theta u) = \cos(\pi\alpha)\|D^\alpha_\theta u\|^2_{L^2}, \tag{6}$$
$$(D^\alpha_\theta u, D^\alpha_{-\theta} v) + (D^\alpha_\theta v, D^\alpha_{-\theta} u) = 2\cos(\pi\alpha)(D^\alpha_\theta u, D^\alpha_\theta v). \tag{7}$$

where $(\cdot, \cdot)$ denote the inner product in $L^2(\mathbb{R}^N)$.

Now, for $\alpha = \frac{\beta+1}{2}$, let

$$X = \{u \in J^\alpha_{M_1}(\mathbb{R}^N) : \ \|u\| = (|u|^2_{M_1} + \|b^{1/2} u\|^2_{L^2})^{\frac{1}{2}} < \infty\},$$

then E is a reflexive and separable Hilbert apace with the inner product

$$\langle u, v \rangle_X = -\int_{|\theta|=1} (D^\alpha_\theta u, D^\alpha_{-\theta} v) M_1(d\theta) + \int_{\mathbb{R}^N} b(x) uv dx. \tag{8}$$

Also, from Lemma 2, we know that if the measure M_1 satisfies (4), then $X \subset H^\alpha(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, $2 \leq p \leq 2^*$. We will denote by $S > 0$ the best Sobolev constant of the embedding $X \subset L^{2^*}(\mathbb{R}^N)$, that is:

$$S = \inf_{u \in J^\alpha_{M_1}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{|\theta|=1} \|D^\alpha_\theta u\|^2_{L^2} M_1(d\theta) + \int_{\mathbb{R}^N} b(x) u^2(x) dx}{\left(\int_{\mathbb{R}^N} |u|^{2^*} dx\right)^{\frac{2}{2^*}}}. \tag{9}$$

Note that by assumption (5), we know if $u \in X$, then $\|D^\alpha_\theta u\|_{L^2}, \|D^\alpha_{-\theta} u\|_{L^2} \in L^2(\mathbb{S}^{N-1}, M_2(\theta))$. For a given $r > 0$, we define

$$f_r(x,t)=\begin{cases} f(x,t), & \text{if } |x|\le r,\\ 0, & \text{if } |x|>r.\end{cases}$$

Let

$$F_r(x,u)=\int_0^u f_r(x,t)dt=\begin{cases} F(x,u), & \text{if } |x|\le r,\\ 0, & \text{if } |x|>r.\end{cases}$$

3 Main result

In this Section we will state our main result and the proof of it relies on the variational principle of Ekeland.

For given $r>0$ and $w\in X$, the functional $J_{w,r}: X\to\mathbb{R}$ corresponding to problem (1) is defined by

$$\begin{aligned} J_{w,r}(u) &= H\left(-\frac{\cos(\pi\alpha)}{2}\int_{|\theta|=1}\|D_\theta^\alpha u\|_{L^2}^2 M_1(d\theta)+\frac{1}{2}\int_{\mathbb{R}^N} b(x)u^2(x)dx\right)\\ &\quad -\int_{\mathbb{R}^N} F_r(x,u(x))dx-\int_{|\theta|=1}(D_{-\theta}^\alpha u, D_\theta^\alpha w)M_2(d\theta)-\frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}dx. \end{aligned}\tag{10}$$

By assumptions, we know $J_{w,r}\in C^1(X,\mathbb{R})$ and using Lemma 3, we have

$$\begin{aligned} J'_{w,r}(u)v &= h\left(-\frac{\cos(\pi\alpha)}{2}\int_{|\theta|=1}\|D_\theta^\alpha u\|_{L^2}^2 M_1(d\theta)+\frac{1}{2}\int_{\mathbb{R}^N} b(x)u^2(x)dx\right)\\ &\quad\times\left(-\int_{|\theta|=1}(D_\theta^\alpha u, D_{-\theta}^\alpha v)M_1(d\theta)+\int_{\mathbb{R}^N} b(x)uvdx\right)\\ &\quad -\int_{|\theta|=1}(D_\theta^\alpha w, D_{-\theta}^\alpha v)M_2(d\theta)-\int_{\mathbb{R}^N} f_r(x,u)vdx-\int_{\mathbb{R}^N}|u|^{2^*-2}uvdx\\ &= h\left(-\frac{\cos(\pi\alpha)}{2}\int_{|\theta|=1}\|D_\theta^\alpha u\|_{L^2}^2 M_1(d\theta)+\frac{1}{2}\int_{\mathbb{R}^N} b(x)u^2(x)dx\right)\\ &\quad\times\left(-\cos(\pi\alpha)\int_{|\theta|=1}(D_\theta^\alpha u, D_\theta^\alpha v)M_1(d\theta)+\int_{\mathbb{R}^N} b(x)uvdx\right)\\ &\quad -\int_{|\theta|=1}(D_\theta^\alpha w, D_{-\theta}^\alpha v)M_2(d\theta)-\int_{\mathbb{R}^N} f_r(x,u)vdx-\int_{\mathbb{R}^N}|u|^{2^*-2}uvdx. \end{aligned}\tag{11}$$

Thus the critical points of $J_{w,r}$ on X are solutions of the problem (1).
Before starting our result, we need the following assumptions:
(B) $b\in C(\mathbb{R}^N,\mathbb{R})$, $\inf_{\mathbb{R}^N} b(x)=b_0>0$ and $\lim_{|x|\to\infty} b(x)=\infty$.

(H1) $h \in L^1(0,r)$, $r > 0$;
(H2) $\limsup_{t\to 0^+} \frac{H(t)}{t^{\alpha_0}} < \infty$ with $0 < \alpha_0 < \frac{2^*}{2}$, where $H(t) = \int_0^t h(s)ds$;
(H3) there exist $0 < \beta_0 < \frac{1}{2}$ and a positive constant C_0 such that

$$H(t) \geq C_0 t^{\beta_0} \quad \text{for all} \;\; t > 0;$$

(F1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and it is a function verifying the following growth condition: there exists a positive constant $c_1 > 0$ such that

$$|f(x,\xi)| \leq c_1(1 + |\xi|^{q-1}), \quad \text{a.e. } x \in \mathbb{R}^N, \; \xi \in \mathbb{R},$$

where $2 < q < 2^*$;
(F2) there exist $A > 0$ and $\sigma > 2\alpha_0$ such that

$$F(x,\xi) \geq A|\xi|^{\sigma}, \quad \text{as} \quad \xi \to 0, \tag{12}$$

with $F(t,x) = \int_0^x f(t,s)ds$.

Therefore by (F1) and (F2), we know, there exist a positive constant $c_1 > 0$, $A > 0$ and $0 < \sigma < 2\alpha_0$ such that

$$\begin{aligned} |f_r(x,\xi)| \leq c_1(1 + |\xi|^{q-1}), \quad \text{for all} \quad |x| \leq r, \;\; 2 < q < 2^*, \\ F_r(x,\xi) \geq A|\xi|^{\sigma}, \quad \text{for all} \quad |x| \leq r, \qquad \text{as} \quad \xi \to 0. \end{aligned}$$

Now, we can state our main result.

Theorem 1. Assume that M_1 and M_2 satisfy the condition (H). Moreover we assume that (B), (H1), (H2), (F1) and (F2) hold, then the problem (1) has a nontrivial solution.

Here, we need some auxiliary lemmas.

Lemma 4. There exist ϱ, ρ such that $J_{w,r} \geq \varrho$ for $\|u\| = \rho$.

Proof. Let R be a positive constant and fix $w \in X$ with $\|w\| \leq R$. Now, for given $r > 0$ and $w \in X$ with $\|w\| \leq R$, from Lemmas 1, 2 and similar method in [13], one can get

$$\begin{aligned} \int_{|\theta|=1} (D^{\alpha}_{-\theta}u, D^{\alpha}_{\theta}v) M_2(d\theta) &\leq \int_{|\theta|=1} \|D^{\alpha}_{-\theta}u\|_{L^2} \|D^{\alpha}_{\theta}v\|_{L^2} M_2(d\theta) \\ &\leq \frac{M_2(\mathbb{S}^{N-1})}{c_{M_1}} \|u\|\|v\|, \end{aligned} \tag{13}$$

for all $u, v \in X$.

Set $\Omega := \{x \in \mathbb{R}^N : \; |x| \leq r\}$, hence, by (H3), Hölder's inequality, (9) and (13), for $\|u\|$ is sufficiently small we have,

$$\begin{aligned}
J_{w,r}(u) &\geq C_0\left(-\frac{\cos(\pi\alpha)}{2}\int_{|\theta|=1}\|D_\theta^\alpha u\|_{L^2}^2 M_1(d\theta)+\frac{1}{2}\int_{\mathbb{R}^N} b(x)u^2(x)dx\right)^{\beta_0}\\
&\quad -\int_{\mathbb{R}^N} F_r(x,u(x))dx-\int_{|\theta|=1}\|D_{-\theta}^\alpha u\|_{L^2}\|D_\theta^\alpha w\|_{L^2}M_2(d\theta)-\frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}dx\\
&\geq C_0\left(-\frac{\cos(\pi\alpha)}{2}\right)^{\beta_0}\|u\|^{2\beta_0}-\int_\Omega F_r(x,u(x))dx\\
&\quad -\int_{|\theta|=1}\|D_{-\theta}^\alpha u\|_{L^2}\|D_\theta^\alpha w\|_{L^2}M_2(d\theta)-\frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*}dx\\
&\geq C_0\left(-\frac{\cos(\pi\alpha)}{2}\right)^{\beta_0}\|u\|^{2\beta_0}-|\Omega|^{\frac{2^*-q}{2^*}}\left(\int_\Omega |u|^{2^*}dx\right)^{\frac{q}{2^*}}\\
&\quad -\frac{RM_2(S^{N-1})}{c_{M_1}}\|u\|-\frac{1}{2^*}S^{-\frac{2^*}{2}}\|u\|^{2^*}+c\\
&\geq C_0\left(-\frac{\cos(\pi\alpha)}{2}\right)^{\beta_0}\|u\|^{2\beta_0}-|\Omega|^{\frac{2^*-q}{2^*}}S^{-\frac{2^*}{2}}\|u\|^q-\frac{RM_2(\mathbb{S}^{N-1})}{c_{M_1}}\|u\|-\frac{1}{2^*}S^{-\frac{2^*}{2}}\|u\|^{2^*}+c,
\end{aligned}$$

since $2\beta_0 < 1 < q < 2^*$, then there exist ϱ, ρ such that $J_{w,r} \geq \varrho$ for $\|u\| = \rho$. ■

Set $\overline{B}_{r_0}(0) = \{u \in X;\ \ \|u\| \leq r_0\}$, then we have the following lemma:

Lemma 5. The functional $J_{w,r}$ is bounded from below in $\overline{B}_{r_0}(0)$; moreover $\overline{J} = \inf_{\overline{B}_{r_0}(0)} J_{w,r} < 0$.

Proof. From the definition of $J_{w,r}$, it is clear that the functional $J_{w,r}$ is bounded from below in $\overline{B}_{r_0}(0)$. Now, let $\varphi \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$ with $\|\varphi\| \leq r_0$, by (H2) and (F2), for $t > 1$, it follows that

$$\begin{aligned}
J_{w,r}(t\varphi) &= H\left(-\frac{t^2\cos(\pi\alpha)}{2}\int_{|\theta|=1}\|D_\theta^\alpha\varphi\|_{L^2}^2 M_1(d\theta)+\frac{t^2}{2}\int_{\mathbb{R}^N} b(x)\varphi^2(x)dx\right)\\
&\quad -\int_{\mathbb{R}^N} F_r(x,t\varphi)dx-t\int_{|\theta|=1}(D_{-\theta}^\alpha\varphi, D_\theta^\alpha w)M_2(d\theta)-\frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N}|\varphi|^{2^*}dx\\
&\leq \frac{t^{2\alpha_0}}{2^{\alpha_0}}\|\varphi\|^{\alpha_0}-At^\sigma\int_{\mathbb{R}^N}|\varphi|^\sigma dx+t\frac{RM_2(\mathbb{S}^{N-1})}{c_{M_1}}\|\varphi\|-\frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N}|\varphi|^{2^*}dx\\
&\leq \frac{t^{2\alpha_0}}{2^{\alpha_0}}r_0^{\alpha_0}-At^\sigma\int_{\mathbb{R}^N}|\varphi|^\sigma dx+t\frac{RM_2(\mathbb{S}^{N-1})}{c_{M_1}}r_0-\frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N}|\varphi|^{2^*}dx\\
&\quad \to -\infty, \quad \text{as } t \to +\infty.
\end{aligned}$$

Thus there exist $t_0 > 1$ such that $J_{w,r}(t_0\varphi) < 0$ ■

Now, using the Ekeland variational principle to $J_{w,r}$ on $\overline{B}_{r_0}(0)$ endowed with distance $\tau(u,\vartheta) = \|u - \vartheta\|$, so there exists a sequence $\{u_{n,w,r}\} \subset \overline{B}_r$ such that:

$$J_{w,r}(u_{n,w,r}) \to \inf_{\overline{B}_r} J_{w,r} = \overline{J},$$

we infer that

$$J_{w,r}(u_{n,w,r}) - J_{w,r}(\vartheta) \le \frac{\|u_{n,w,r} - \vartheta\|}{n},$$

for all $\vartheta \neq u_{n,w,r}$. Since $J_{w,r}$ is of class C^1 then

$$J'_{w,r}(u_{n,w,r}) \to 0$$

and thus we have

$$J_{w,r}(u_{n,w,r}) \to \overline{J} \quad \text{and} \quad J'_{w,r}(u_{n,w,r}) \to 0,$$

which yields that $\{u_{n,w,r}\}$ is a $(\text{P.S})_{\overline{J}}$ sequence to $J_{w,r}$. Since $\{u_{n,w,r}\}$ is a $(\text{P.S})_{\overline{J}}$ sequence and using the definition of $J_{w,r}$, there exists a constant $C_4 > 0$ such that

$$\|u_{n,w,r}\| \le C_4, \quad \forall n \in \mathbb{N}. \tag{14}$$

So passing to a subsequence it necessary, it can be assumed that $\{u_{n,w,r}\}$ converges weakly to $u_{w,r}$ in X and thus $\{u_{n,w,r}\}$ converges strongly to $u_{w,r}$ in $L^r_{\text{loc}}(\mathbb{R}^N)$, $2 \le r < 2^*$ and $|u_{n,w,r}|^{2^*-2}u_{n,w,r} \to |u_{w,r}|^{2^*-2}u_{w,r}$ in $L^{\frac{2^*}{2^*-q}}(\mathbb{R}^N)$. As $J_{w,r}$ is of class C^1 then

$$(J'_{w,r}(u_{n,w,r}) - J'_{w,r}(u_{w,r}))v \to 0, \quad \text{as} \quad n \to +\infty,$$

for any $v \in X$.

Hence it remains to prove that $u_{w,r} \neq 0$. It is well known that $J_{w,r}(u_{n,w,r}) \to \overline{J}$ then by (9) and (13), one can get

$$\begin{aligned} \overline{J} + o(1) &= J_{n,w,r}(u_{n,w,r}) \\ &\ge C_0\left(-\frac{\cos(\pi\alpha)}{2}\right)^{\beta_0} \|u_{n,w,r}\|^{2\beta_0} - |\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{2^*}{2}} \|u_{n,w,r}\|^q \\ &\quad -\frac{RM_2(\mathbb{S}^{N-1})}{c_{M_1}}\|u_{n,w,r}\| - \frac{1}{2^*}S^{-\frac{2^*}{2}}\|u_{n,w,r}\|^{2^*}. \end{aligned}$$

It follows that

$$|\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{2^*}{2}} \|u_{w,r}\|^q + \frac{RM_2(\mathbb{S}^{N-1})}{c_{M_1}}\|u_{w,r}\| + \frac{1}{2^*}S^{-\frac{2^*}{2}}\|u_{w,r}\|^{2^*} > -\overline{J} + o(1),$$

which implies that

$$|\Omega|^{\frac{2^*-q}{2^*}} S^{-\frac{2^*}{2}} \|u_{w,r}\|^q + \frac{RM_2(\mathbb{S}^{N-1})}{c_{M_1}}\|u_{w,r}\| + \frac{1}{2^*}S^{-\frac{2^*}{2}}\|u_{w,r}\|^{2^*} > -\overline{J} > 0,$$

consequently $u_{w,r} \neq 0$.

From the previous lemmas and by applying the Ekeland principle, the problem (1) has a nontrivial solution.

■

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